

# Characterizing mixing and measurement in quantum mechanics

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What fundamental constraints characterize the relationship between a mixture  $\rho = \sum_i p_i \rho_i$  of quantum states, the states  $\rho_i$  being mixed, and the probabilities  $p_i$ ? What fundamental constraints characterize the relationship between prior and posterior states in a quantum measurement? In this paper we show that there are many surprisingly strong constraints on these mixing and measurement processes that can be expressed simply in terms of the eigenvalues of the quantum states involved. These constraints capture in a succinct fashion what it means to say that a quantum measurement *acquires* information about the system being measured, and considerably simplify the proofs of many results about entanglement transformation.

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## I. INTRODUCTION

Quantum mechanics harbours a rich structure whose investigation and explication is the goal of quantum information science [1,2]. At present only a limited understanding of the fundamental static and dynamic properties of quantum information has been obtained, and many major problems remain open. In particular, we would like a detailed ontology and quantitative methods of description for the different types of information and dynamical processes possible within quantum mechanics. An example of the pursuit of these goals along a specific line of thought has been the partial development of a theory of entangled quantum states; see for example the work in [3–12].

The purpose of the present paper is to pose and partially solve two fundamental problems about the static and dynamic properties of quantum information. The first of these problems is to characterize the process of *mixing* quantum states. More precisely, if  $\rho = \sum_i p_i \rho_i$  is a mixture of quantum states  $\rho_i$  with probabilities  $p_i$ , what constraints relate the properties of  $\rho$  to the probability distribution  $p_i$  and the quantum states  $\rho_i$ ? The second problem is to characterize the relationship between the prior and posterior states in a quantum measurement. The result of our investigations is a set of two *static constraints* on mixtures of quantum states, two *dynamic constraints* on the quantum measurement process, and two partial *converse* results, one to the static constraints, and the other to the dynamic constraints. The statement of each of these results is rather easily understood, so we review the statements now, before proceeding to the proofs

and consequences in the main body of the paper.

Suppose we mix a set of quantum states  $\rho_i$  according to the probability distribution  $p_i$ . Then we will show that this mixing process must satisfy the constraint equations:

$$\lambda \left( \sum_i p_i \rho_i \right) \prec \sum_i p_i \lambda(\rho_i) \quad (1)$$

$$\bigoplus_i p_i \lambda(\rho_i) \prec \lambda \left( \sum_i p_i \rho_i \right). \quad (2)$$

In these equations the notation  $\oplus$  denotes a direct sum of vectors,  $\lambda(X)$  denotes the vector of eigenvalues of the matrix  $X$  arranged so the components appear in non-increasing order, and the relation “ $\prec$ ” is the *majorization* relation<sup>1</sup>. As an example of the notation used in (2), suppose  $p_1 = 1/3, p_2 = 2/3, \rho_1 = \text{diag}(3/4, 1/4)$  and  $\rho_2 = \text{diag}(1/5, 4/5)$ . Then Equation (2) becomes

$$\frac{1}{3} \begin{bmatrix} \frac{3}{4} \\ \frac{1}{4} \end{bmatrix} \bigoplus \frac{2}{3} \begin{bmatrix} \frac{4}{5} \\ \frac{1}{5} \end{bmatrix} \prec \lambda \left( \frac{1}{3} \begin{bmatrix} \frac{3}{4} & 0 \\ 0 & \frac{1}{4} \end{bmatrix} + \frac{2}{3} \begin{bmatrix} \frac{1}{5} & 0 \\ 0 & \frac{4}{5} \end{bmatrix} \right), \quad (3)$$

which is equivalent to

$$\begin{bmatrix} \frac{1}{4} \\ \frac{1}{12} \\ \frac{1}{8} \\ \frac{1}{15} \\ \frac{2}{15} \end{bmatrix} \prec \begin{bmatrix} \frac{37}{60} \\ \frac{23}{60} \\ 0 \\ 0 \end{bmatrix}. \quad (4)$$

A formal definition of majorization appears in Subsection II B, however for now the essential intuition to grasp is that the relation  $x \prec y$  means that the vector  $x$  is more “mixed” (or “disordered”) than  $y$ . Thus, Equation (1)

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<sup>1</sup>Note that the vectors on the left and right hand sides in (2) may be of different dimension; in such cases we extend whichever vector is of lesser dimension by padding it with zero entries, to enable comparison using the majorization relation.

captures the intuition that  $\sum_i p_i \rho_i$  is more mixed, on average, than the states  $\rho_i$  appearing in the ensemble. The intuition behind (2) is a little more complex. Imagine that we prepare the state  $\rho$  by randomly choosing a value for  $i$  according to the probability distribution  $p_i$ , and then preparing the corresponding state  $\rho_i$ . Our quantum state, including a description of  $i$ , may be written as  $\sum_i p_i |i\rangle\langle i| \otimes \rho_i$ . We then “throw away” the state  $|i\rangle$  representing our random choice of  $i$ , leaving only the state  $\sum_i p_i \rho_i$ . The relation (2) expresses the fact that when we throw away  $i$ , the state of the quantum system becomes less disordered.

Suppose we perform a measurement on a quantum mechanical system initially in the state  $\rho$ , obtaining measurement result  $i$  with probability  $p_i$ , and corresponding posterior state  $\rho'_i$ . What constraints are placed on the relationship between  $\rho$ ,  $p_i$  and  $\rho'_i$ ? We will show that the following two *dynamic constraints* must be satisfied:

$$\lambda(\rho) \prec \sum_i p_i \lambda(\rho'_i) \quad (5)$$

$$\bigoplus_i p_i \lambda(\rho'_i) \prec \lambda(\rho). \quad (6)$$

The intuition behind (5) is that quantum measurements acquire information about the state of the system being measured, and thus after measurement the state of the system is less mixed, on average, than before. The intuition behind (6) is a little more complex, but can be understood using Zurek’s approach [13] to decoherence and quantum measurement. Recall that in this approach a measurement involves three systems: the system being measured, which starts in the state  $\rho$ , and ends in the state  $\rho'_i$ ; a measuring device, which starts in some standard state, and finishes in a “pointer state”  $|i\rangle$  recording the result of the measurement, and an environment which “decoheres” the measuring device, ensuring that it behaves in an essentially classical fashion. The system and measuring device interact unitarily during the measurement, ensuring that there is no change in the amount of disorder present in the system. The subsequent environmental decoherence process can also be thought of as a type of measurement, in which the different outcomes are averaged over. In this view, the environment continually measures the state of the measuring apparatus, resulting in a final state  $\sum_i p_i |i\rangle\langle i| \otimes \rho'_i$  for the measuring apparatus and system being measured. This decoherence process causes an increase in the disorder present in the system, which is the intuition behind (6). More succinctly, (6) may be thought of as capturing the notion that the total ensemble of possible quantum states is more disordered after a measurement than it is before.

The importance of the static constraints (1)-(2) and the dynamics constraints (5)-(6) is further reinforced by the fact that in each case there is a type of converse to these equations. In this introduction we focus only on

the more interesting case of the converse to the dynamic constraints (5) and (6), however rather similar remarks hold also for the static constraints (1) and (2). Suppose  $p_i$  is a probability distribution, and  $\rho$  and  $\rho'_i$  are quantum states such that

$$\lambda(\rho) \prec \sum_i p_i \lambda(\rho'_i). \quad (7)$$

Then we will show that there exists a quantum measurement whose measurement outcomes may be labelled by a pair of indices  $(i, j)$ , such that for any fixed  $i$  and for all  $j$  the posterior state of the quantum system after measurement is  $\rho'_i$ , and the probabilities  $p_{ij}$  for the  $(i, j)$ th measurement outcome satisfy  $\sum_j p_{ij} = p_i$ . Unfortunately, this result is not a tight converse to equations (5) and (6), due to the introduction of the extra index  $j$ , however for many purposes it is a sufficiently strong converse. We will show that even the equations (5) and (6) together do not completely characterize the quantum measurement process, however I believe it likely that there is a simple characterization of the measurement process along similar lines that may be expressed entirely in terms of the eigenvalues of the prior and posterior states, and the probabilities of the different measurement outcomes. Of course, it is true that the quantum measurement formalism already provides such a characterization, in the form of a matrix equation, however equations such as (5) and (6) provide far more explicit information, and as such, are likely to be more useful in practice. We will demonstrate the utility of this approach by application to the problem of entanglement transformation, simplifying the proofs of several known results about entanglement transformation [4,7–9,14].

There is a striking level of symmetry in the equations (1)-(2), (5)-(6), which we will also see in the partial converse results. It is obviously tempting to suggest that this reflects some deeper underlying principle, much as Maxwell’s equations may be derived from a deeper action principle based on the Faraday tensor, or the still deeper principles of gauge invariance and relativity. Unfortunately, I have not yet succeeding in obtaining a satisfactory form for such a deeper principle. Presumably, such a deeper principle might assist in tightening the partial converse results, or perhaps tightening the partial converses may shed light on the origin of Equations (1)-(2), (5)-(6).

In explaining the intuitive meanings of the equations (1)-(2) and (5)-(6) we have used language such as the “disorder” present in a quantum state. One might wonder if it is possible to write down entropic statements capturing these intuitions. We will show that each of these equations in fact implies an entropic statement whose content corresponds to the intuition we have described. Of course, entropic statements should really only be interpreted in the asymptotic limit where we have a large number of identical copies of a system available;

the advantage of Equations (1)-(2) and (5)-(6) is that they are stronger forms of these asymptotic statements which may be applied to single quantum systems.

This paper contains six fundamental results (together with a number of applications), expressed in the four constraint equations, (1)-(2), (5)-(6), and the partial converses to (1)-(2) and (5)-(6). We now review antecedents of these results in the existing literature. Equation (1) is an elementary consequence of classic results in the theory of majorization. Equation (2) follows as a corollary of work of Uhlmann [15], Ruskai (unpublished, 1993) and Nielsen [16] on the relationship between mixed states and probability distributions. Equations (5) and (6) are implicit in the work of Vidal [8] on entanglement transformation, and the partial converse to (5)-(6) is implicit in the work of Jonathan and Plenio [9] on entanglement transformation, building on earlier work by Nielsen [4]. A proof of Equation (5) in the context of entanglement transformation has also been previously obtained by Jonathan, Nielsen, Schumacher and Vidal (unpublished, 1999). There are several advantages to the point of view taken in the present paper. First, measurement is in some sense a more fundamental process than entanglement transformation, and Equations (5) and (6) highlight the fundamental connection between measurement and majorization for the first time, incidentally explaining why there is a connection between entanglement transformation and majorization: it arises as a result of a deeper connection between measurement and majorization. Second, the proofs in the present paper are novel, and have the advantage of proceeding from a more unified point of view than earlier work. As a result they are, perhaps, more elegant and informative than earlier proofs, especially the proof of the partial converse to (5)-(6), which is a substantial improvement of and extension to existing constructions. Several other items of related work are also worth pointing out. There is a substantial mathematical literature on the problem of characterizing the properties of sums  $A + B$  of Hermitian matrices  $A$  and  $B$ , and Fulton [17] has written a nice review of recent progress on this problem, which is closely related to the problem of mixing of density matrices. Hardy [14] has introduced techniques in the context of entanglement transformation that can be used to prove (5) and the partial converse to (5)-(6). Fuchs and Jacobs (unpublished, 2000) have obtained a beautiful and quite different proof of (5), after hearing of the result from Nielsen. Finally, the procedure described in this paper to prove the partial converse to (5)-(6) is a generalization of the procedures for entanglement transformation for pure states found by Nielsen in [4], and subsequently improved in independent work by Hardy, Jonathan and Nielsen (described in Chapter 12 of [1]), by Jensen and Schack [18], and by Werner (unpublished, 2000).

The paper is structured as follows. We begin in Section II by reviewing the two main tools that will be used

in this paper, the theory of generalized measurements in quantum mechanics, and the mathematical theory of majorization. Section III contains proofs of the static constraints (1) and (2) on the mixing of quantum states, and the dynamic constraints (5) and (6) on quantum measurement, and explores some elementary consequences of these results. In Section IV we prove the partial converses to (1)-(2) and (5)-(6). Section V explains how the results of the present paper may be used to obtain simplified proofs of known results about entanglement transformation. Finally, Section VI concludes the paper with a discussion of some open problems and future directions.

## II. GENERALIZED MEASUREMENTS AND MAJORIZATION

Before proceeding to the main results of the paper it is useful to first review some background material on generalized measurements and the mathematical theory of majorization. All discussion in this and succeeding sections is to be understood in the context of finite-dimensional vector spaces, although infinite-dimensional modifications seem likely to hold, perhaps with some technical modifications.

### A. Generalized measurements

In this paper we use the *generalized measurements formalism* as our basic tool for the description of quantum measurements. The theory of generalized quantum measurements is an extension of the projective measurements described in most quantum mechanics textbooks. The reason the generalized measurements formalism is adopted is because it is better adapted to the description of many realistic quantum measurement schemes. However, it is important to appreciate that the generalized measurement formalism follows from standard quantum mechanics, in the sense that any generalized measurement can be understood as arising from the combination of unitary evolution and a projective measurement, a correspondence made explicit below. Nevertheless, the formalism of generalized measurements is in many ways more useful and mathematically elegant than the standard formulation of quantum measurement in terms of projectors. More detailed introductions to the theory of generalized measurements may be found in [19,20,1,21].

Mathematically, a generalized measurement is specified by a set  $\{E_i\}$  of *measurement matrices* satisfying the *completeness relation*  $\sum_i E_i^\dagger E_i = I$ . The index  $i$  on the measurement matrices is in one-to-one correspondence with the possible outcomes that may occur in the measurement. The rule used to connect the measurement

matrices to physics is that if the prior state of the quantum system is  $\rho$  then the outcome  $i$  occurs with probability  $p_i = \text{tr}(E_i \rho E_i^\dagger)$ , and the posterior state is given by  $\rho'_i = E_i \rho E_i^\dagger / \text{tr}(E_i \rho E_i^\dagger)$ .

Generalized measurements are obviously more general than the projective measurements described in most textbooks. Projective measurements have the feature that they are *repeatable*, in the sense that if one performs a projective measurement twice in a row on a quantum system, then one will obtain the same result both times. By contrast, most real measurements don't have this feature of being repeatable, which tips us off to the need for the formalism of generalized measurements. Nevertheless, even the generalized measurement formalism can be understood in terms of projective measurements as follows: the effect of a generalized measurement on a quantum system is *equivalent* to a unitary interaction between the system being measured and another "ancilla" system, followed by a projective measurement on the ancilla system. More precisely, suppose  $\{E_i\}$  is a set of measurement matrices satisfying the completeness relation  $\sum_i E_i^\dagger E_i = I$ . We introduce an ancilla system with orthonormal basis elements  $|i\rangle$  indexed by the possible measurement outcomes. Define a matrix  $U$  acting on the joint quantum system-ancilla by the action:

$$U|\psi\rangle|0\rangle \equiv \sum_i E_i |\psi\rangle|i\rangle, \quad (8)$$

where  $|0\rangle$  is some standard state of the ancilla and  $|\psi\rangle$  is an arbitrary state of the quantum system being measured. It is easy to show using the completeness relation  $\sum_i E_i^\dagger E_i = I$  that  $U$  can be extended to a unitary matrix acting on the entire state space of the joint system. Suppose we perform the unitary transformation  $U$  on the joint quantum system-ancilla, and then do a projective measurement of the ancilla in the  $|i\rangle$  basis. It is then easily checked that the result of the measurement is  $i$  with probability  $p_i = \text{tr}(E_i \rho E_i^\dagger)$  and the corresponding post-measurement state of the system is  $\rho'_i = E_i \rho E_i^\dagger / \text{tr}(E_i \rho E_i^\dagger)$ . Thus, the effect on the quantum system is exactly as we have described above for a generalized quantum measurement. Conversely, it is not difficult to verify that the effect of a unitary interaction between system and ancilla followed by a projective measurement on the ancilla can always be understood in terms of a generalized measurement (see for example Chapter 8 of [1]).

## B. Majorization

Our primary tool in the study of mixing and measurement in quantum mechanics is the theory of majorization, whose basic elements we now review. The following review only covers elementary aspects of the theory of

majorization, and the reader is referred to Chapters 2 and 3 of [22], [23] or [24] for more extensive background.

The basic motivation for majorization is to capture what it means to say that one probability distribution is "more mixed" than another. Suppose  $x = (x_1, \dots, x_d)$  and  $y = (y_1, \dots, y_d)$  are two  $d$ -dimensional real vectors; we usually suppose in addition that  $x$  and  $y$  are probability distributions, that is, the components are non-negative and sum to one, but the following definitions apply in the case of general  $x$  and  $y$  as well. The relation  $x \prec y$ , read " $x$  is majorized by  $y$ ", is intended to capture the notion that  $x$  is more mixed (i.e. disordered) than  $y$ . To make the formal definition, we introduce the notation  $\downarrow$  to denote the components of a vector rearranged into non-increasing order, so  $x^\downarrow = (x_1^\downarrow, \dots, x_d^\downarrow)$ , where  $x_1^\downarrow \geq x_2^\downarrow \geq \dots \geq x_d^\downarrow$ . We say that  $x$  is majorized by  $y$  and write  $x \prec y$ , if

$$\sum_{j=1}^k x_j^\downarrow \leq \sum_{j=1}^k y_j^\downarrow, \quad (9)$$

for  $k = 1, \dots, d-1$ , and with the inequality holding with equality when  $k = d$ .

It is perhaps not so clear how this definition connects with any natural notion of comparative disorder. We will state but not prove a remarkable result connecting majorization to a natural notion of mixing. It can be shown (see Chapter 2 of [22]) that  $x \prec y$  if and only if  $x = \sum_i p_i P_i y$ , where the  $p_i$ s form a probability distribution and the  $P_i$ s are permutation matrices. Thus, when  $x \prec y$  we can imagine that  $y$  is the input probability distribution to a noisy channel which randomly permutes the symbols sent through the channel, inducing an output probability distribution  $x$ . From this characterization many other important results follow with minimal effort; for example, it can easily be shown that if  $x \prec y$  then the Shannon entropy of the distribution  $x$  must be at least as great as that of  $y$ .

The connection between majorization and quantum mechanics arises primarily as a result of *Horn's lemma* (proved in [25]; for a simple proof see [16]), which states that  $x \prec y$  if and only if there exists a unitary matrix  $u = (u_{ij})$  such that  $x_i = \sum_j |u_{ij}|^2 y_j$ . This fundamental relationship between majorization and unitarity ensures many close connections between majorization and quantum mechanics.

As an elementary consequence of Horn's lemma we have *Ky Fan's maximum principle*, which states that for any Hermitian matrix  $A$ , the sum of the  $k$  largest eigenvalues of  $A$  is the maximum value of  $\text{tr}(AP)$ , where the maximum is taken over all  $k$ -dimensional projectors  $P$ ,

$$\sum_{j=1}^k \lambda_j(A) = \max_P \text{tr}(AP). \quad (10)$$

To see this, note that choosing  $P$  to be the projector onto the space spanned by the  $k$  eigenvectors of  $A$  with the  $k$  largest eigenvalues results in  $\text{tr}(AP) = \sum_{j=1}^k \lambda_j(A)$ . The proof of Ky Fan's maximum principle will be completed if we can show that  $\text{tr}(AP) \leq \sum_{j=1}^k \lambda_j(A)$  for any  $k$ -dimensional projector  $P$ . To see this, let  $|e_1\rangle, \dots, |e_d\rangle$  be an orthonormal basis chosen such that  $P = \sum_{j=1}^k |e_k\rangle\langle e_k|$ . Let  $|f_1\rangle, \dots, |f_d\rangle$  be an orthonormal set of eigenvectors for  $A$ , ordered so the corresponding eigenvalues are in non-increasing order. Then

$$\langle e_j | A | e_j \rangle = \sum_{k=1}^d |u_{jk}|^2 \lambda_k(A), \quad (11)$$

where  $u_{jk} \equiv \langle e_j | f_k \rangle$  is unitary. By Horn's lemma it follows that  $(\langle e_j | A | e_j \rangle) \prec \lambda(A)$ , which implies that

$$\text{tr}(AP) = \sum_{j=1}^k \langle e_j | A | e_j \rangle \leq \sum_{j=1}^k \lambda_j(A), \quad (12)$$

as required.

Ky Fan's maximum principle gives rise to a useful constraint on the eigenvalues of a *sum* of two Hermitian matrices, that  $\lambda(A+B) \prec \lambda(A) + \lambda(B)$ . To see this, choose a  $k$ -dimensional projector  $P$  such that

$$\sum_{j=1}^k \lambda_j(A+B) = \text{tr}((A+B)P) \quad (13)$$

$$= \text{tr}(AP) + \text{tr}(BP) \quad (14)$$

$$\leq \sum_{j=1}^k \lambda_j(A) + \sum_{j=1}^k \lambda_j(B), \quad (15)$$

where the last line also follows from Ky Fan's maximum principle.

Another consequence of Horn's lemma is that given a density matrix  $\rho$  and a probability distribution  $p_i$  there exist pure states  $|\psi_i\rangle$  such that  $\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$  if and only if  $(p_i) \prec \lambda(\rho)$  (see [16,15]; this result was also obtained in unpublished work by Ruskai (1993)), where it is understood that if the vector  $(p_i)$  contains more terms than the vector  $\lambda(\rho)$  then the vector  $\lambda(\rho)$  is to be “padded” with extra zero terms. The proof of this result is simply to combine Horn's lemma with the classification of ensembles  $\{p_i, |\psi_i\rangle\}$  consistent with a given density matrix  $\rho$ , as discovered independently by Schrödinger [26], Jaynes [27] and Hughston, Jozsa and Wootters [28]. See [16] for the details of the proof.

This notion of “padding” vectors of unequal dimension so they can be compared by the majorization relation is surprisingly useful, and we adopt the general convention that when  $x$  and  $y$  are of different dimension then  $x \prec y$  means that  $\tilde{x} \prec \tilde{y}$ , where  $\tilde{x}$  and  $\tilde{y}$  are padded with extra zero components to ensure that they have the same dimension. For example,  $(1/3, 1/3, 1/3) \prec (1/2, 1/2, 0)$  since

$(1/3, 1/3, 1/3) \prec (1/2, 1/2, 0)$ . It is easy to check that this extended notion of majorization is well-defined, provided  $x$  and  $y$  both have non-negative components, and this will be the case for all the applications in this paper. Similarly, it is often useful to write  $x = y$  provided the padded versions of  $x$  and  $y$  are equal, that is, the non-zero entries of  $x$  and  $y$  are equal. With these conventions, it is easy to see that algebraic manipulations proceed exactly as one would expect. For example, for non-negative real vectors  $w, x, y, z$  if  $w \prec x, x = y, y \prec z$  then obviously  $w \prec z$ , even if all four vectors have different dimensionality. We occasionally make use of such elementary observations in proofs, without explicit comment.

The final result about majorization we shall need is that if  $P_i$  are a set of orthogonal projectors such that  $\sum_i P_i = I$ , and  $\rho$  is a density matrix, then [22]

$$\lambda \left( \sum_i P_i \rho P_i \right) \prec \lambda(\rho). \quad (16)$$

Intuitively, if a projective measurement of a quantum system is performed, but we do not learn the result of the measurement, then the state of the system after measurement is more mixed than it was before. One way of proving this relation is via Horn's lemma; a sketch follows. First, note that it suffices to prove that  $\lambda(P\rho P + Q\rho Q) \prec \lambda(\rho)$ , where  $P$  and  $Q = I - P$  are two orthogonal projectors satisfying  $P + Q = I$ . Once this is proved, the general relation (16) follows by a simple induction. However, if we define a unitary matrix  $U \equiv P - Q$  then it is easy to verify that

$$P\rho P + Q\rho Q = \frac{\rho + U\rho U^\dagger}{2}. \quad (17)$$

Applying Horn's lemma and the easily proved fact that if  $x_1 \prec y$  and  $x_2 \prec y$  then  $(x_1 + x_2)/2 \prec y$ , it follows with a little simple linear algebra that  $\lambda(P\rho P + Q\rho Q) \prec \lambda(\rho)$ .

### III. PROOF OF CONSTRAINTS ON MIXING AND MEASUREMENT IN QUANTUM MECHANICS

In this section we prove the four constraints, (1)-(2), (5)-(6). The first and second of these are static constraints on the mixing of quantum states, proved in Subsection III A. The third and fourth constraint equations are dynamic constraints on the quantum measurement process, proved in Subsection III B. Finally, some simple consequences of these results are discussed in Subsection III C.

### A. Static constraints on mixing quantum states

**Theorem 1:** Suppose  $\rho = \sum_i p_i \rho_i$  is a convex combination of quantum states  $\rho_i$  with probabilities  $p_i$ . Then

$$\lambda(\rho) \prec \sum_i p_i \lambda(\rho_i) \quad (18)$$

$$\bigoplus_i p_i \lambda(\rho_i) \prec \lambda(\rho). \quad (19)$$

**Proof of (18):** This is an immediate consequence of the fact that  $\lambda(A + B) \prec \lambda(A) + \lambda(B)$  for any two Hermitian matrices  $A$  and  $B$ , as proved in Subsection II B.

**Proof of (19):** As noted in Subsection II B, if a density matrix  $\rho$  can be written as a convex combination of pure states  $|\psi_i\rangle$ ,  $\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$ , then it follows that  $(p_i) \prec \lambda(\rho)$ , where  $(p_i)$  denotes the vector whose entries are the probabilities  $p_i$ . Equation (19) is a corollary of this result. To see this, note that if  $r_{ij}$  are the eigenvalues of  $\rho_i$  and  $|i, j\rangle$  the corresponding orthonormal eigenvectors then (19) is equivalent to the equation

$$(p_i r_{ij}) \prec \lambda(\rho), \quad (20)$$

which follows from the results of Subsection II B and the observation that

$$\rho = \sum_i p_i \rho_i = \sum_{ij} p_i r_{ij} |i, j\rangle\langle i, j|. \quad (21)$$

This completes the proof of Theorem 1. ■

### B. Dynamical constraints on quantum measurement

**Theorem 2:** Suppose  $\{E_i\}$  is a set of measurement matrices satisfying the completeness relation  $\sum_i E_i^\dagger E_i = I$ . Then the quantum measurement described by these matrices must satisfy the following four constraints:

$$\lambda\left(\sum_i E_i \rho E_i^\dagger\right) \prec \sum_i \lambda(E_i \rho E_i^\dagger) \quad (22)$$

$$\bigoplus_i \lambda(E_i \rho E_i^\dagger) \prec \lambda\left(\sum_i E_i \rho E_i^\dagger\right) \quad (23)$$

$$\lambda(\rho) \prec \sum_i \lambda(E_i \rho E_i^\dagger) \quad (24)$$

$$\bigoplus_i \lambda(E_i \rho E_i^\dagger) \prec \lambda(\rho). \quad (25)$$

A slightly different way of stating Theorem 2 is to define  $p_i$  to be the probability of obtaining outcome  $i$  when the measurement defined by the matrices  $\{E_i\}$  is performed on the system, and let  $\rho'_i = E_i \rho E_i^\dagger / \text{tr}(E_i \rho E_i^\dagger)$  be the corresponding posterior states. Then the following four equations are equivalent to (22)-(25):

$$\lambda\left(\sum_i p_i \rho'_i\right) \prec \sum_i p_i \lambda(\rho'_i) \quad (26)$$

$$\bigoplus_i p_i \lambda(\rho'_i) \prec \lambda\left(\sum_i p_i \rho'_i\right) \quad (27)$$

$$\lambda(\rho) \prec \sum_i p_i \lambda(\rho'_i) \quad (28)$$

$$\bigoplus_i p_i \lambda(\rho'_i) \prec \lambda(\rho). \quad (29)$$

Theorem 2 is a fundamental constraint on the dynamics that may occur during a quantum measurement. Equations (26) and (27) are, of course, merely the dynamical expression of the static constraints found earlier in Theorem 1. Equations (28) and (29) represent novel constraints of an essentially dynamical nature, connecting as they do the prior and posterior states of the quantum measurement. Intuitively, Equation (28) captures the notion that a quantum measurement “gains information” (on average) about a quantum state, since it says that the eigenvalues of the initial state  $\rho$  are, on average, more disordered than the eigenvalues of the posterior states  $\rho'_i$ . Intuitively, the second dynamic constraint, (29) captures the notion that the *total ensemble* of possible quantum states is more disordered after the measurement than before. Thus, (28) and (29) represent complementary constraints on the evolution of a quantum system during a quantum measurement process.

The constraints (26)-(29) are applicable even for very complex measurement processes. For example, a single mode cavity undergoing direct photodetection by an ideal photodetector can be described by a special case of the generalized measurements formalism known as the *quantum trajectories* or *stochastic Schrödinger equation* picture (see [29,30] for a review and references). In this picture, if the system is started in the state  $\rho$  then the final state of the system is  $\rho_h$ , where “ $h$ ” is used here to denote not just a single measurement outcome, but rather the complete history recorded by the photodetector, that is, all the times at which photocounts occurred. Then (28) and (29) may be written as

$$\lambda(\rho) \prec \int d\mu(h) \lambda(\rho_h) \quad (30)$$

$$\bigoplus_h d\mu(h) \lambda(\rho_h) \prec \lambda(\rho), \quad (31)$$

where the integral is a functional integral over all possible photodetection histories, and  $d\mu(h)$  is the corresponding measure on histories.

**Proof of Theorem 2:** The first two equations of Theorem 2, (22) and (23), are immediate consequences of the deeper *static* constraints on quantum mechanics introduced in Theorem 1; here we are merely enumerating

the implications these static constraints have for dynamics. The remaining constraints, (24) and (25), are genuine quantum dynamical constraints relating the prior and posterior states of a quantum measurement.

**Proof of (24):** Suppose  $\rho$  is a positive matrix which can be written in the block form:

$$\rho = \begin{bmatrix} A & X \\ X^\dagger & B \end{bmatrix}. \quad (32)$$

For our purposes  $\rho$  will most often be a density matrix (and thus satisfy  $\text{tr}(\rho) = 1$ ), but the results we prove hold for a general positive matrix. We will show that  $\lambda(\rho) \prec \lambda(A) + \lambda(B)$ . (Recall our conventions on padding, which imply that the vectors of eigenvalues for  $A$  and  $B$  are to be extended by zeroes in such a way that they contain as many entries as the vector of eigenvalues of  $\rho$ ).  $\rho$  is a positive matrix, so there must exist a matrix  $D = [D_1|D_2]$  such that  $\rho = D^\dagger D$ , where the matrices  $D_1$  and  $D_2$  have the same number of columns as  $A$  and  $B$ , respectively, and both have the same number of rows as  $\rho$ . Thus we have

$$\begin{bmatrix} A & X \\ X^\dagger & B \end{bmatrix} = D^\dagger D = \begin{bmatrix} D_1^\dagger D_1 & D_1^\dagger D_2 \\ D_2^\dagger D_1 & D_2^\dagger D_2 \end{bmatrix}, \quad (33)$$

from which we read off  $A = D_1^\dagger D_1$  and  $B = D_2^\dagger D_2$ . Using the results of Subsection II B and the fact that the eigenvalues of a product  $EF$  of matrices  $E$  and  $F$  are the same as the eigenvalues of  $FE$ , up to padding by zeroes, we see that

$$\lambda(\rho) = \lambda(D^\dagger D) \quad (34)$$

$$= \lambda(DD^\dagger) \quad (35)$$

$$= \lambda(D_1 D_1^\dagger + D_2 D_2^\dagger) \quad (36)$$

$$\prec \lambda(D_1 D_1^\dagger) + \lambda(D_2 D_2^\dagger) \quad (37)$$

$$= \lambda(D_1^\dagger D_1) + \lambda(D_2^\dagger D_2) \quad (38)$$

$$= \lambda(A) + \lambda(B), \quad (39)$$

and thus  $\lambda(\rho) \prec \lambda(A) + \lambda(B)$ , as claimed. This method for eliminating off-diagonal block terms was introduced by Wielandt to connect the Weyl and Aronszajn inequalities (cited as [31] in Chapter 3 of [22].)

As a straightforward consequence we see by induction that for any positive matrix  $\rho$  and complete set of orthogonal projectors  $\{P_i\}$ :

$$\lambda(\rho) \prec \sum_i \lambda(P_i \rho P_i) \quad (40)$$

Extending even further, suppose  $\{E_i\}$  is any set of measurement matrices defining a generalized measurement, and  $\rho$  is a positive matrix. As in Subsection II A we can introduce an ancilla system with an orthonormal basis  $|i\rangle$  in one-to-one correspondence with the indices on the measurement matrices  $E_i$  and define a unitary matrix  $U$  which has the action

$$U|\psi\rangle|0\rangle = \sum_i E_i |\psi\rangle|i\rangle, \quad (41)$$

where  $|0\rangle$  is some standard state of the ancilla. Then we have  $\lambda(\rho) = \lambda(\rho \otimes |0\rangle\langle 0|)$ , since the non-zero eigenvalues of  $\rho$  and  $\rho \otimes |0\rangle\langle 0|$  are the same. Simple algebra and (40) imply that

$$\lambda(\rho) = \lambda(U(\rho \otimes |0\rangle\langle 0|)U^\dagger) \quad (42)$$

$$\prec \sum_i \lambda((I \otimes |i\rangle\langle i|)U(\rho \otimes |0\rangle\langle 0|)U^\dagger(I \otimes |i\rangle\langle i|)) \quad (43)$$

$$= \sum_i \lambda(E_i \rho E_i^\dagger \otimes |i\rangle\langle i|) \quad (44)$$

$$= \sum_i \lambda(E_i \rho E_i^\dagger), \quad (45)$$

where in the last line we used  $\lambda(E_i \rho E_i^\dagger \otimes |i\rangle\langle i|) = \lambda(E_i \rho E_i^\dagger)$ , since the non-zero entries agree. This completes the proof of (24).

**Proof of (25):** Again, let  $U$  be the unitary matrix constructed in Subsection II A to implement the measurement described by the measurement matrices  $\{E_i\}$ , namely, any unitary matrix having the action

$$U|\psi\rangle|0\rangle = \sum_i E_i |\psi\rangle|i\rangle. \quad (46)$$

Again, we have  $\lambda(\rho) = \lambda(\rho \otimes |0\rangle\langle 0|)$ , since the non-zero eigenvalues of  $\rho$  are the same as those of  $\rho \otimes |0\rangle\langle 0|$ , and thus  $\lambda(\rho) = \lambda(U(\rho \otimes |0\rangle\langle 0|)U^\dagger)$ . It follows from Equation (16) that

$$\lambda \left( \sum_i (I \otimes |i\rangle\langle i|)U(\rho \otimes |0\rangle\langle 0|)U^\dagger(I \otimes |i\rangle\langle i|) \right) \prec \lambda(\rho), \quad (47)$$

and thus

$$\lambda \left( \sum_i E_i \rho E_i^\dagger \otimes |i\rangle\langle i| \right) \prec \lambda(\rho). \quad (48)$$

This last equation is obviously equivalent to the statement we set out to prove,  $\blacksquare$

$$\bigoplus_i \lambda(E_i \rho E_i^\dagger) \prec \lambda(\rho), \quad (49)$$

which concludes the proof of Theorem 2.  $\blacksquare$

### C. Consequences of the constraint equations

The constraints proved in Theorems 1 and 2 are very strong and, not surprisingly, have many interesting consequences. We now elucidate a few of these consequences using the notions of *Schur-concavity* and *Schur-convexity*.

A Schur-convex function  $f(\cdot)$  is a real-valued function which preserves the majorization relation, in the sense that if  $x \prec y$  then  $f(x) \leq f(y)$ . Simple necessary and sufficient conditions for a function to be Schur-convex are known [22], and many interesting functions are Schur-convex. These include, for example, the function  $x \rightarrow f(x) \equiv \sum_{j=1}^d x_j^k$ , for any  $k \geq 1$ . Similarly, a *Schur-concave* function  $f(\cdot)$  is one such that if  $x \prec y$  then  $f(x) \geq f(y)$ . Equivalently,  $f(\cdot)$  is Schur-concave if  $-f(\cdot)$  is Schur-convex. Perhaps the canonical example of a Schur-concave function is the Shannon entropy  $H(x) = -\sum_j x_j \log_2(x_j)$ , so that whenever  $x \prec y$  it follows that  $H(x) \geq H(y)$ , giving further justification to the intuitive notion that  $x \prec y$  means that  $x$  is more disordered than  $y$ . Applying the Schur-concavity of Shannon's entropy to the results of Theorems 1 and 2 we obtain an attractive suite of results. First, applying the Schur-concavity of  $H(\cdot)$  to (18) gives

$$S(\rho) \geq H \left( \sum_i p_i \lambda(\rho_i) \right). \quad (50)$$

Applying the concavity of the Shannon entropy to the right hand side, we obtain as a corollary the concavity of the von Neumann entropy,

$$S(\rho) \geq \sum_i p_i S(\rho_i). \quad (51)$$

Applying the Schur-concavity of  $H(\cdot)$  to (19) and doing some simple algebra gives

$$\sum_i p_i S(\rho_i) + H(p_i) \geq S(\rho). \quad (52)$$

This result was obtained previously by Lanford and Robinson [32] using different techniques. Applying the Schur-concavity of  $H(\cdot)$  to (28), followed by the concavity of the Shannon entropy, gives

$$S(\rho) \geq \sum_i p_i S(\rho'_i). \quad (53)$$

Essentially the same result has been obtained previously in the context of entanglement transformation [3], where it expresses the fact that local processes cannot increase the amount of entanglement present in a system. Finally, applying the Schur-concavity of  $H(\cdot)$  to (29) gives the beautiful inequality

$$H(p_i) + \sum_i p_i S(\rho'_i) \geq S(\rho), \quad (54)$$

which implies that in order to lower the entropy of a system by an amount  $\Delta$ , on average, the information  $H(p_i)$  collected by the measurement must be at least as large as  $\Delta$ . This fact can be seen as a quantum mechanical expression of the principle, expressed by Landauer

[33] and fleshed out by Bennett [34] and Zurek [35], that measurement of a physical system carries with it a thermodynamic cost when the measurement record is erased, and proper accounting of this cost enables one to solve the conundrum posed by Maxwell's demon. (See [36] for a review.)

Applying the Schur-convexity of the functions  $f(x) = \sum_i x_i^k$  for  $k \geq 1$  to the results of Theorems 1 and 2 also give a number of interesting constraints. The arguments used are analogous to those given above for the Shannon entropy, so the details will be omitted, and we merely state the results:

$$\sum_i p_i^k \text{tr}(\rho_i^k) \leq \text{tr}(\rho^k) \leq \sum_i p_i \text{tr}(\rho_i^k) \quad (55)$$

$$\sum_i p_i^k \text{tr}((\rho'_i)^k) \leq \text{tr}(\rho^k) \leq \sum_i p_i \text{tr}((\rho'_i)^k). \quad (56)$$

#### IV. PARTIAL CONVERSES TO THE CONSTRAINTS ON MIXING AND MEASUREMENT

Given the constraints on mixing and measurement described in Theorems 1 and 2 it is natural to ask if these constraints completely *characterize* the processes of mixing and measurement, respectively. We will show below that the answer to this question is *no*. However, partial progress towards achieving simple characterizations of mixing and measurement may be reported in the form of a partial converse to Theorem 1, described below in Subsection IV A, and a partial converse to Theorem 2, described in Subsection IV B.

##### A. Partial converse to the constraints on mixing

Given the constraints Theorem 1 imposes on mixing it is natural to ask whether these constraints completely characterize the mixing process. That is, given a density matrix  $\rho$ , probabilities  $p_i$  and vectors  $\lambda_i$  with non-negative, non-increasing components which sum to one, and such that

$$\lambda(\rho) \prec \sum_i p_i \lambda_i \quad (57)$$

$$\bigoplus_i p_i \lambda_i \prec \lambda(\rho), \quad (58)$$

does it follow that there exist density matrices  $\rho_i$  such that  $\lambda(\rho_i) = \lambda_i$  and  $\rho = \sum_i p_i \rho_i$ ?

We will show below that the answer to this question is no, however I suspect that some characterization along similar lines is possible. Progress towards such a characterization can be reported in the form of a partial converse to Theorem 1, which states that provided (57)

holds then there exist states  $\rho_{ij}$  and a probability distribution  $p_{ij}$  such that  $\lambda(\rho_{ij}) = \lambda_i$ , independent of the value of the index  $j$ , and  $p_i = \sum_j p_{ij}$  for each  $i$ , as well as  $\rho = \sum_{ij} p_{ij} \rho_{ij}$ . That is, in order to obtain a converse to (57) we need to introduce an extra index,  $j$ . We will show below that it is necessary to introduce the extra index if only (57) is assumed as a hypothesis for the converse. Let's state and prove the partial converse as Theorem 3.

**Theorem 3:** Suppose  $\rho$  is a density matrix and  $\lambda_i$  are vectors with non-negative, non-increasing components summing to one. Suppose  $p_i$  are probabilities such that

$$\lambda(\rho) \prec \sum_i p_i \lambda_i. \quad (59)$$

Then there exist density matrices  $\rho_{ij}$  and a probability distribution  $p_{ij}$  such that  $p_i = \sum_j p_{ij}$ ,  $\lambda(\rho_{ij}) = \lambda_i$ , and  $\rho = \sum_{ij} p_{ij} \rho_{ij}$ .

To prove Theorem 3 we need the result stated in Subsection II B that  $x \prec y$  if and only if there exist probabilities  $q_j$  and permutation matrices  $P_j$  such that  $x = \sum_j q_j P_j y$ . Applying this result with the assumption (59) we obtain

$$\lambda(\rho) = \sum_{ij} p_i q_j P_j \lambda_i. \quad (60)$$

Working in the basis in which  $\rho$  is diagonal, and defining  $\Lambda_i$  to be the diagonal matrix with diagonal entries  $\lambda_i$ , we may set  $p_{ij} \equiv p_i q_j$  and  $\rho_{ij} \equiv P_j \Lambda_i P_j^\dagger$ , obtaining  $p_i = \sum_j p_{ij}$  and  $\lambda(\rho_{ij}) = \lambda_i$ . Finally, the equation  $\rho = \sum_{ij} p_{ij} \rho_{ij}$  follows immediately from these definition and (60), completing the proof.

What of a tight converse to Theorem 1? It is easy to see that it is not possible to obtain a tight converse to (57) alone, as follows. Suppose we choose  $\rho = I/2$  to be the completely mixed state of a single qubit, and define a probability distribution on just one outcome, the trivial distribution  $p_1 = 1$ , with corresponding vector  $\lambda_1 = (1, 0)$ . Clearly,  $\lambda(\rho) \prec \sum_i p_i \lambda_i$ , yet it is not possible to find a state  $\rho_1$  such that  $\rho = p_1 \rho_1$  and  $\lambda(\rho_1) = \lambda_1$ . Thus, in this example, it is necessary to introduce extra indices, just as was done in Theorem 3.

Might it be that conditions (57) and (58) together completely characterize the mixing process? The following example, due to Julia Kempe, shows that this is not the case. Suppose we consider a qubit system, and choose  $\rho = \text{diag}(5/12, 7/12)$ ,  $p_1 = p_2 = 1/2$ , and  $\lambda_1 = (1, 0)$ ,  $\lambda_2 = (1/2, 1/2)$ . It is easy to verify that conditions (57) and (58) are satisfied with these choices. Unfortunately, it is not possible to find states  $\rho_1$  and  $\rho_2$  with vectors of eigenvalues  $\lambda_1$  and  $\lambda_2$  such that  $\rho = p_1 \rho_1 + p_2 \rho_2$ , since with these choices for  $\lambda_1$  and  $\lambda_2$  it follows that  $\rho_1$  must be a pure state and  $\rho_2 = I/2$  the completely mixed state, so  $p_1 \rho_1 + p_2 \rho_2$  has eigenvalues  $3/4$  and  $1/4$ , which are not equal to  $5/12$  and  $7/12$ .

Despite this example, I believe it likely that conditions along the lines of (57) and (58) may be used to completely characterize the process of mixing in quantum mechanics.

## B. Partial converse to the constraints on measurement

Given the constraints Theorem 2 imposes on the quantum measurement process it is natural to ask whether these constraints completely characterize the possible posterior states and probabilities which may occur in such a measurement? That is, supposing  $\rho$  is a density matrix,  $p_i$  is a probability distribution, and  $\rho'_i$  are density matrices such that

$$\lambda(\rho) \prec \sum_i p_i \lambda(\rho'_i) \quad (61)$$

$$\bigoplus_i p_i \lambda(\rho'_i) \prec \lambda(\rho), \quad (62)$$

does it follow that there exist measurement matrices  $\{E_i\}$  satisfying the completeness relation  $\sum_i E_i^\dagger E_i = I$  and giving the states  $\rho'_i$  as posterior states, with probabilities  $p_i$ , when the measurement is performed on a system initially prepared in the state  $\rho$ ?

We will show below that the answer to this question is no, however I suspect that some characterization along similar lines is possible. Progress towards such a characterization can be reported in the form of a partial converse to Theorem 2, which states that provided the relation (61) holds, then there is a quantum measurement described by measurement matrices  $\{E_{ij}\}$  such that the corresponding posterior states  $\rho'_{ij}$  satisfy  $\rho'_{ij} = \rho_i$  for every  $j$ , and the measurement probabilities  $p_{ij}$  satisfy  $\sum_j p_{ij} = p_i$ . Thus, in order to obtain a converse to (61) we need to introduce an extra index,  $j$ , just as we did earlier in the partial converse to Theorem 1. Also analogously to that case, we show below that it is necessary to introduce the extra index with only (61) as hypothesis for the converse. Let's state and prove the partial converse as Theorem 4.

**Theorem 4:** Suppose  $\rho$  is a density matrix with vector of eigenvalues  $\lambda$ , and  $\sigma_i$  are density matrices with vectors of eigenvalues  $\lambda_i$ . Suppose  $p_i$  are probabilities such that

$$\lambda \prec \sum_i p_i \lambda_i \quad (63)$$

Then there exist matrices  $\{E_{ij}\}$  and a probability distribution  $p_{ij}$  such that

$$\sum_{ij} E_{ij}^\dagger E_{ij} = I \quad (64)$$

$$E_{ij} \rho E_{ij}^\dagger = p_{ij} \sigma_i \quad (65)$$

$$\sum_j p_{ij} = p_i. \quad (66)$$

To prove Theorem 4, we again use the result that  $x \prec y$  if and only if there exist probabilities  $q_j$  and permutation matrices  $P_j$  such that  $x = \sum_j q_j P_j y$ . By assumption we have  $\lambda \prec \sum_i p_i \lambda_i$  and thus there exist permutation matrices  $P_j$  and probabilities  $q_j$  such that

$$\lambda = \sum_{ij} p_i q_j P_j \lambda_i. \quad (67)$$

Without loss of generality we may assume that  $\rho$  and  $\sigma_i$  are all diagonal in the same basis, with non-increasing diagonal entries, since if this is not the case then it is an easy matter to prepend or append unitary matrices to the measurement matrices to obtain the correct transformation. With this convention, we define matrices  $E_{ij}$  by

$$E_{ij} \sqrt{\rho} \equiv \sqrt{p_i q_j} \sqrt{\sigma_i} P_j^\dagger. \quad (68)$$

In order for  $E_{ij}$  to be well-defined by this formula alone it is necessary that  $\rho$  be invertible. If this is not the case then the  $E_{ij}$  are defined on the support of  $\rho$  by the formula (68), and to act as the zero operator on the orthocomplement of the support of  $\rho$ . It is convenient to let  $P$  be the projector onto the support of  $\rho$ . Note that we have

$$\sqrt{\rho} \left( \sum_{ij} E_{ij}^\dagger E_{ij} \right) \sqrt{\rho} = \sum_{ij} p_i q_j P_j \sigma_i P_j^\dagger. \quad (69)$$

Comparing with (67) we see that the right-hand side of the last equation is just  $\rho$  and thus

$$\sqrt{\rho} \left( \sum_{ij} E_{ij}^\dagger E_{ij} \right) \sqrt{\rho} = \rho, \quad (70)$$

from which we deduce that  $\sum_{ij} E_{ij}^\dagger E_{ij} = P$ , the projector onto the support of  $\rho$ . Letting  $Q \equiv I - P$  be the projector onto the orthocomplement of the support, we can append an additional measurement matrix  $E_{00} \equiv Q$  to the collection  $E_{ij}$  to ensure that the completeness relation  $\sum_{ij} E_{ij}^\dagger E_{ij} = I$  is satisfied. Furthermore, from the definition (68) it follows that

$$E_{ij} \rho E_{ij}^\dagger = p_i q_j \sigma_i, \quad (71)$$

and thus upon performing a measurement defined by the measurement matrices  $\{E_{ij}\}$  the result  $(i, j)$  occurs with probability  $p_{ij} = p_i q_j$ ,  $\sum_j p_{ij} = p_i$ , and the post-measurement state is  $\sigma_i$ . This completes the proof of Theorem 4.

Theorem 4 is not a sharp converse to the condition of Equation (61) because of the extra index  $j$ . Introducing some such index is certainly necessary with the present hypotheses, as may be seen by considering an example

with  $\lambda = (1/2, 1/2)$ , and the trivial probability distribution on one outcome,  $p_1 = 1$ , with  $\lambda_1 = (1, 0)$ . Then  $\lambda \prec p_1 \lambda_1$ , but it is clear that there does not exist an  $E_1$  such that  $E_1 \rho E_1^\dagger = \rho_1$ , where  $\lambda(\rho) = \lambda$ ,  $\lambda(\rho_1) = \lambda_1$  and  $E_1^\dagger E_1 = I$ , because the last equation implies that  $E_1$  must be unitary. It is not difficult to construct more complex examples to convince oneself that this behaviour is generic.

Might it be that the conditions (61) and (62) together characterize the posterior states and probabilities achievable through a quantum measurement? The following argument, due to Julia Kempe and the author, shows that this is not the case. Suppose we consider a qubit system, and choose  $\rho = \text{diag}(5/12, 7/12)$ ,  $p_1 = p_2 = 1/2$ , and  $\rho'_1 = \text{diag}(1, 0)$ ,  $\rho'_2 = \text{diag}(1/2, 1/2)$ . It is easy to verify that conditions (61) and (62) are satisfied with these choices. Unfortunately, it is not possible to find measurement matrices  $E_1$  and  $E_2$  satisfying  $\sum_i E_i^\dagger E_i = I$  and giving posterior states  $\rho'_1$  and  $\rho'_2$  with equal probabilities  $1/2$ , when the state  $\rho$  is measured. This can be seen in a variety of ways. A simple direct way is to note that the purity of  $\rho'_1$  implies that  $E_1$  must have the form  $E_1 = \alpha |a\rangle\langle b|$  for normalized states  $|a\rangle$  and  $|b\rangle$ , and some  $\alpha > 0$ . Thus

$$E_2^\dagger E_2 = I - E_1^\dagger E_1 \quad (72)$$

$$= I - \alpha^2 |b\rangle\langle b| \quad (73)$$

$$= (1 - \alpha^2) |b\rangle\langle b| + |c\rangle\langle c|, \quad (74)$$

where  $|c\rangle$  is orthonormal to  $|b\rangle$ . The polar decomposition gives  $E_2 = U \sqrt{E_2^\dagger E_2}$  for some unitary  $U$ , so

$$E_2 = \sqrt{1 - \alpha^2} U |b\rangle\langle b| + U |c\rangle\langle c|. \quad (75)$$

We are requiring that  $E_2 \rho E_2^\dagger = I/4$ , so it must be the case that  $E_2$  is non-singular, and thus  $\alpha < 1$ . Premultiplying by  $E_2^{-1}$  and postmultiplying by  $(E_2^\dagger)^{-1}$  gives

$$\rho = \frac{1}{4(1 - \alpha^2)} |b\rangle\langle b| + \frac{1}{4} |c\rangle\langle c|. \quad (76)$$

Since  $|b\rangle$  and  $|c\rangle$  are orthonormal it follows that such a  $\rho$  cannot be equal to  $\text{diag}(5/12, 7/12)$ , which is the desired contradiction. Despite this example, I believe it likely that conditions along the lines of (61) and (62) may be used to characterize the process of measurement in quantum mechanics.

## V. ENTANGLEMENT TRANSFORMATION

The problem of *entanglement transformation* is a natural context in which the results of the present paper may be applied. The problem of entanglement transformation arises as a consequence of the fundamental question of how may we convert one type of physical resource into another, and there has been considerable effort devoted

to determining when it is possible to convert one type of entanglement to another. In [4] a connection was noted between entanglement transformation and majorization, namely, that if  $|\psi\rangle$  and  $|\phi\rangle$  are pure states of a bipartite quantum system with components belonging to Alice ( $A$ ) and Bob ( $B$ ) respectively, then Alice and Bob can transform the state  $|\psi\rangle$  into the state  $|\phi\rangle$  using local operations on their respective systems and classical communication between Alice and Bob, if and only if

$$\lambda_\psi \prec \lambda_\phi, \quad (77)$$

where  $\lambda_\psi$  (respectively  $\lambda_\phi$ ) is the vector of eigenvalues of the reduced density matrix for Alice's system when the joint system is in the state  $|\psi\rangle$  ( $|\phi\rangle$ ). As per usual, the components of such vectors are ordered into non-increasing order. This result has subsequently been generalized by Vidal [7] to the case of conclusive transformation, and even further by Jonathan and Plenio [9] to the problem where Alice and Bob are supplied with a state  $|\psi\rangle$  and wish to transform this state into an *ensemble* of states in which the state  $|\phi_i\rangle$  occurs with probability  $p_i$ . (See also Hardy [14] for an instructive alternative approach to results of this type.) The necessary and sufficient condition for such a transformation to be possible is that [9]:

$$\lambda_\psi \prec \sum_i p_i \lambda_{\phi_i}. \quad (78)$$

We now explain how this result can be seen as an easy consequence of the results proved in the present paper, and thus the connection between majorization and entanglement is really a consequence of a deeper connection between majorization and measurement. By a result of Lo and Popescu [37], it is possible to transform  $|\psi\rangle$  into the ensemble  $\{p_i, |\phi_i\rangle\}$  by local operations and classical communication if and only if it is possible to make the transformation via the following simplified procedure: first, Alice performs a generalized measurement on her state, then sends the result to Bob, who performs a unitary operation on his system conditional on the outcome of the measurement Alice made. Let  $\rho = \text{tr}_B(|\psi\rangle\langle\psi|)$  be the initial state of Alice's system, and suppose Alice performs a quantum measurement described by measurement matrices  $E_i$ , so that outcome  $i$  occurs with probability  $p_i$  and  $(E_i \otimes U_i)|\psi\rangle = \sqrt{p_i}|\phi_i\rangle$ , for some unitary operator  $U_i$  acting on Bob's system. Considering Alice's system alone and observing that that  $E_i \rho E_i^\dagger = \sigma_i$ , where  $\sigma_i = p_i \text{tr}(|\phi_i\rangle\langle\phi_i|)$ , we deduce from Theorem 2 that

$$\lambda_\rho \prec \sum_i p_i \lambda_{\sigma_i}, \quad (79)$$

which is equivalent to (78). To prove the converse, suppose (78) holds. Then by Theorem 4 there exists a quantum measurement described by measurement matrices  $E_{ij}$ , and probabilities  $p_{ij}$  such that

$$E_{ij} \rho E_{ij}^\dagger = p_{ij} \sigma_i; \quad \sum_j p_{ij} = p_i. \quad (80)$$

The procedure for Alice and Bob to produce the ensemble is for Alice to perform the measurement described by the set  $E_{ij}$ . The post-measurement state  $|\phi_{ij}\rangle$  is then a purification [1] of the state  $\sigma_i$ , and it can be shown (see [28] or Section 2.5 of [1]) that by performing an appropriate unitary transformation Bob can convert the state  $|\phi_{ij}\rangle$  into the state  $|\phi_i\rangle$ , with total probability  $p_i$  of obtaining the state  $|\phi_i\rangle$ . Thus Equation (78) represents a necessary and sufficient condition for it to be possible to transform the state  $|\psi\rangle$  into the ensemble  $\{p_i, |\phi_i\rangle\}$  by local operations and classical communication.

## VI. CONCLUSION

We have shown that there are strong fundamental constraints on the processes of mixing and measurement in quantum mechanics that may be naturally expressed in the language of majorization. Although the results in the present paper don't completely characterize these processes, they suggest that there may exist a simple set of conditions which substantially simplify the usual characterization of these processes via operator equations. Another interesting direction for further research is to generalize the constraints on measurements obtained in this paper to better understand how two or more states may transform simultaneously under a measurement. Once again, although this problem is in principle already "solved", in the sense that there is an operator equation specifying exactly what transformations may occur, results such as those in the present paper and in [38] indicate that much more explicit characterizations may be possible. Such explicit conditions are likely to have applications to fundamental problems such as the problem of transformation of mixed state entanglement [3], and to the problem of determining to what extent the acquisition of information about the identity of a quantum state disturbs the system being measured [39].

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